GENERAL PHYSICS LABORATORY MANUAL

(Physics I – Physics II)

Introduction to Experimental Physics

Measurement, Errors, and Data Analysis

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1. Introduction

1.1. Physical Quantities and Measurement

Physics is an experimental science, and experiments are based on the measurement of physical quantities. A **physical quantity** is any property of a physical system that can be measured and expressed numerically. Examples include length, mass, time, temperature, electric current, and voltage. In laboratory work, these quantities are measured using appropriate instruments and are reported together with their units.

A **measurement** consists of comparing an unknown quantity with a standard unit. Because all measuring instruments have limited precision, no measurement can be perfectly exact. For this reason, every measured value is associated with an uncertainty. Understanding what a measurement represents, how it is expressed, and how its uncertainty arises is a fundamental part of experimental physics.

In this laboratory course, emphasis is placed on using standard units, performing measurements carefully, and interpreting measured values correctly. The aim is not only to obtain numerical results, but also to understand their physical meaning and limitations. Proper measurement techniques form the foundation for data analysis, graphical representation, and comparison with theoretical predictions.

Physical Quantities

A physical quantity is a measurable property of a physical system. It is expressed by a numerical value together with a unit and is generally associated with an uncertainty arising from the measurement process. For example, the length of a rod may be measured as $(2.35 \pm 0.01)\,\mathrm{m}$, and the time interval of an event may be measured as $(4.2 \pm 0.1)\,\mathrm{s}$.

Physical quantities can be classified as fundamental or derived. Fundamental quantities, such as length, mass, and time, are defined independently. Derived quantities are defined in terms of fundamental quantities through physical relationships. In practice, derived quantities may be obtained either by calculation from measured fundamental quantities or measured directly using appropriately calibrated instruments.

In laboratory experiments, identifying the relevant physical quantities and expressing them correctly, together with their associated units and uncertainties, is an essential first step for meaningful data analysis.

SI Units

In experimental physics, measurements are reported using the International System of Units (SI), which provides a consistent and universally accepted framework for expressing physical quantities. The use of SI units ensures that experimental results can be clearly communicated and compared across different experiments and laboratories.

The SI system is based on a set of fundamental units, including the meter (m) for length, kilogram (kg) for mass, second (s) for time, ampere (A) for electric current, and kelvin (K) for temperature. Other physical quantities are expressed using derived units that are formed from these base units. For example, velocity is expressed in meters per second (m/s), and force is expressed in newtons (N).

In this laboratory course, all measured and calculated quantities should be expressed in SI units unless otherwise stated. When necessary, measured values obtained in other unit systems must be converted into SI units before further analysis.

Table 1: SI fundamental physical quantities and their base units

Physical Quantity	SI Unit	Unit Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Table 2 presents electromagnetic physical quantities along with their symbols and corresponding SI units. Table 3 summarizes the fundamental physical constants used in electromagnetism. The listed constants include the elementary charge, the rest masses of the electron, proton, and neutron, as well as the permittivity and permeability of free space.

Table 2: Symbols and SI Units of Electromagnetic Physical Quantities

Quantity	Symbol	SI Unit
Mass	m	kg
Electric charge	q or e	C
Linear charge density	λ	C/m
Surface charge density	σ	C/m^2
Volume charge density	ρ	C/m^3
Voltage (potential difference)	V	V
Electric current	I	A
Current density	$\vec{J}$	A/m^2
Resistance	R	Ω
Conductivity	σ	$1/(\Omega\text{m})$
Capacitance	C	F
Dielectric constant	κ	–
Electric field	$\vec{E}$	V/m
Electric flux	Φ	V·m
Electromotive force	ε	V
Electric dipole moment	$\vec{p}$	C·m
Magnetic field	$\vec{B}$	T
Magnetic flux	Φ_m	Wb
Magnetic dipole moment	$\vec{\mu}$	N·m/T
Permeability of free space	μ_0	N/A^2
Permittivity of free space	ε_0	C^2/(N·m^2)
Energy	E, U, K	J
Force	F	N
Frequency	f	Hz
Power	P	W

These constants are essential for understanding electric and magnetic fields, electromagnetic waves, and the formulation of Maxwell's equations.

Table 3: Fundamental Physical Constants

Constant	Value
Elementary charge, e	$1.60217733 \times 10^{-19}$ C
Rest mass of the electron	$9.1093897 \times 10^{-31}$ kg
Rest mass of the proton	$1.6726231 \times 10^{-27}$ kg
Rest mass of the neutron	$1.6749286 \times 10^{-27}$ kg
Permeability of free space, μ_0	$4\pi \times 10^{-7}$ T^2 m^3/J
Permittivity of free space, ε_0	$8.854187817 \times 10^{-12}$ C^2/(J m)

Derived Quantities

Derived quantities are physical quantities that are defined in terms of fundamental quantities through physical relationships. They describe many important properties encountered in laboratory experiments and physical phenomena.

Examples of derived quantities include velocity, acceleration, force, energy, and electric potential difference. These quantities are expressed using units formed from combinations of SI fundamental units. For instance, velocity is expressed in meters per second (m/s), force in newtons (N), and energy in joules (J).

As an example, consider the measurement of velocity. If the distance traveled by an object is measured using a meter stick and the time interval is measured using a stopwatch, the velocity is obtained from these two measured quantities. The resulting value of the velocity therefore depends on both the distance and time measurements. Any uncertainty in the measurement of distance or time contributes to the uncertainty in the calculated velocity.

In general, since derived quantities depend on one or more measured quantities, their uncertainties originate from the uncertainties of the quantities used to obtain them, whether through direct measurement or calculation. Proper consideration of these uncertainties is essential for meaningful data analysis and reliable comparison with theoretical predictions.

1.2. Measurement Instruments and Precision

Experimental measurements are performed using instruments designed to measure specific physical quantities. Common measurement instruments used in physics laboratories include rulers and meter sticks for length, stopwatches for time, balances for mass, thermometers for temperature, and electrical meters for current and voltage. Each instrument is suitable only for certain types of measurements and within a limited range.

All measurement instruments have a finite precision, which limits how accurately a quantity can be measured. Precision is determined by the design of the instrument and is often related to the smallest scale division or digital display increment. For example, a ruler marked in millimeters cannot reliably measure length differences smaller than one millimeter, and a digital stopwatch displaying two decimal places cannot resolve time intervals smaller than its display resolution.

The choice of an appropriate instrument is an important part of experimental design. An instrument with insufficient precision may lead to large uncertainties, while an unnecessarily precise instrument may be impractical or difficult to use. Therefore, selecting an instrument that matches the scale and nature of the measurement is essential for obtaining reliable experimental results.

As shown in Fig. 1, the meter stick scale consists of equally spaced markings. Each major tick corresponds to a length of 1 cm, while the smaller intermediate ticks represent subdivisions of 1 mm. The smallest marked division therefore determines the resolution of the measurement.

Figure 1: Meter stick scale showing major (1 cm) and minor (1 mm) divisions used for length measurements.

It is important to distinguish between precision and accuracy in measurements. Precision refers to the degree of reproducibility of repeated measurements and is largely determined by the resolution of the measuring instrument. For the meter stick shown in Fig. 1, the smallest scale division sets the limit on how precisely a length can be read.

Accuracy, on the other hand, describes how close a measured value is to the true or accepted value of the quantity. A measurement may be precise but not accurate if the instrument is improperly calibrated or if a systematic error is present. Conversely, a measurement may be accurate but have low precision if it is affected by large random variations.

The difference between precision and accuracy can be illustrated with simple laboratory examples. A meter stick may allow repeated length readings with a resolution of 1 mm, resulting in highly consistent measurements. However, if the meter stick expands or contracts due to temperature changes, all measurements may be systematically shifted, leading to precise but inaccurate results.

Similarly, a digital stopwatch may display time intervals with a resolution of 0.01 s, suggesting high precision. Nevertheless, delays caused by human reaction time when starting and stopping the stopwatch can introduce a systematic offset, reducing the accuracy of the measured time interval even when repeated measurements appear consistent.

The opposite situation can also occur, where a measurement is accurate but not precise. For example, a high-quality clock or wristwatch may be well calibrated and display the correct time on average. However, if the clock has a coarse display or updates only in large time steps, it may not allow fine resolution of short time intervals. In this case, the measured time values may be close to the true value (accurate) but lack reproducibility at small scales (low precision).

2. Errors and Uncertainty

All experimental measurements are subject to uncertainty. This uncertainty arises from limitations of measuring instruments, environmental conditions, and the measurement procedure itself. As a result, no measured value can be regarded as exact. Understanding the nature of measurement errors and how they affect experimental results is a fundamental aspect of laboratory work.

In experimental physics, the term *error* does not imply a mistake or incorrect procedure. Instead, it refers to the unavoidable deviation of a measured value from the true or accepted value of a physical quantity. The associated *uncertainty* provides an estimate of the range within which the true value is expected to lie.

Measurement errors are commonly classified into two main categories: systematic errors and random (statistical) errors. These two types of errors influence experimental results in different ways and must be treated differently in data analysis.

2.1. Systematic Errors

Systematic errors are errors that consistently affect measurements in the same direction. They are often caused by factors such as instrument miscalibration, environmental influences, or imperfections in the experimental setup. Because systematic errors introduce a consistent bias, repeated measurements do not eliminate them.

Examples of systematic errors have already been encountered in the discussion of measurement instruments. As illustrated by the meter stick example in the preceding section (see Fig. 1), thermal expansion or contraction of the scale can cause all length measurements to be systematically larger or smaller than the true value. Similarly, as discussed for stopwatch measurements, a consistent delay due to human reaction time can shift all measured time intervals by approximately the same amount.

Such effects do not increase the scatter of repeated measurements but instead shift the measured values away from the true or accepted value. For this reason, systematic errors primarily affect the *accuracy* of a measurement.

2.2. Random (Statistical) Errors

Random errors arise from unpredictable variations in the measurement process. These variations may be caused by fluctuations in experimental conditions, limitations of human perception, or inherent noise in measuring instruments. Unlike systematic errors, random errors cause measurements to scatter around a mean value.

For instance, repeated timing of an event with a stopwatch may yield slightly different results due to variations in reaction time. Random errors affect the *precision* of measurements and can be reduced by repeating measurements and using appropriate averaging techniques. When a measurement is repeated many times under similar conditions, the individual results tend to cluster around a central value, and the average of these measurements provides a more reliable estimate of the quantity being measured.

2.3. Uncertainty in Measurements

The uncertainty associated with a measurement provides a quantitative indication of its reliability. It reflects the combined effects of both systematic and random errors. In laboratory reports, measured values should always be presented together with their associated uncertainties.

Proper estimation and reporting of uncertainty allow experimental results to be meaningfully compared with theoretical predictions and with results obtained by others. In this laboratory course, emphasis is placed on understanding the sources of uncertainty and on reporting measurements in a clear and consistent manner.

In the ruler reading shown in Fig. 2, the indicated position lies between the 4.3 cm and 4.4 cm markings and is visibly closer to 4.3 cm. At a basic level, all observers should agree that the length is approximately 4.3 cm. A conservative way to report this reading is therefore

$$x = 4.3 \pm 0.1 \, \text{cm},$$

which reflects the instrument's smallest marked division and avoids any subjective interpolation.

With more careful observation, one may estimate the position within the smallest scale division. Since the smallest division of the meter stick is 1 mm (0.1 cm), a commonly adopted rule for analog scales is to assign an uncertainty of about half this value. A refined and recommended report is

$$x = 4.35 \pm 0.05 \, \text{cm}.$$

This expression makes use of the available visual information while remaining consistent with the resolution of the measuring instrument and the limitations of human observation. Please note that both readings are consistent with each other when their associated uncertainties are taken into account.
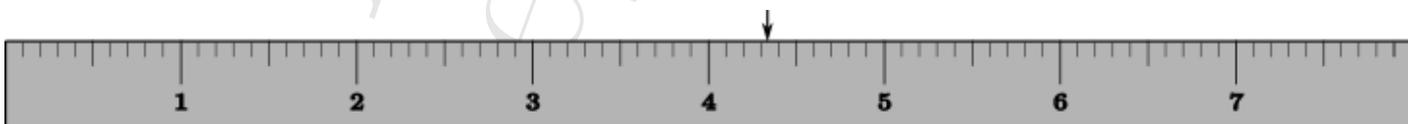

Figure 2: Example of a length measurement illustrating reading uncertainty. The arrow marks a position between two adjacent scale divisions.

For digital measuring instruments, the interpretation of measurements and uncertainties follows a slightly different but more restrictive convention. A digital instrument displays discrete numerical values, and the smallest change shown on the display represents the instrument's resolution.

In this laboratory course, the uncertainty of a digital measurement will be taken to be equal to the resolution of the instrument, unless otherwise stated. For example, if a digital stopwatch displays time intervals with a resolution of 0.01 s, a measured time should be reported as

$$t = 2.34 \pm 0.01 \, \text{s}.$$

In this case, the last displayed digit already reflects the limit of precision, and no additional interpolation between displayed values is justified.

Unlike analogd analog scales, digital instruments do not allow visual estimation within the smallest division. Therefore, reporting uncertainties smaller than the display resolution, or adding extra digits beyond those shown by the instrument, is not acceptable.

In addition to absolute uncertainty, it is often useful to consider the uncertainty of a measurement relative to the size of the measured value itself. This is known as the *fractional uncertainty*. It provides a dimensionless measure of the quality of a measurement and allows uncertainties of different quantities to be compared directly.

The fractional uncertainty is defined as the ratio of the absolute uncertainty to the measured value,

$$\text{fractional uncertainty} = \frac{\Delta x}{x},$$

where x is the measured value and Δx is its absolute uncertainty. Because both numerator and denominator have the same units, the fractional uncertainty is unitless.

For example, a length reported as

$$x = 4.35 \pm 0.05 \, \text{cm}$$

has a fractional uncertainty

$$\frac{\Delta x}{x} = \frac{0.05}{4.35} \approx 0.011.$$

This indicates that the uncertainty is about one percent of the measured value.

Fractional uncertainty is closely related to *relative uncertainty*, which is the same quantity expressed as a percentage,

$$\text{relative uncertainty} = \frac{\Delta x}{x} \times 100\%.$$

In the example above, the relative uncertainty is approximately 1.1%. The same measurement may also be expressed using a relative (percentage) uncertainty. For example, the result

$$x = 4.35 \pm 1.1\%$$

indicates that the uncertainty is 1.1% of the measured value.

Fractional and relative uncertainties are particularly useful when comparing measurements of different magnitudes or when assessing which measured quantity contributes most strongly to the uncertainty of a calculated result. In general, a smaller fractional uncertainty indicates a more precise measurement, regardless of the absolute size of the quantity.

2.4. Significant Figures

Significant figures are used to indicate the precision of a measured or calculated quantity. They reflect the reliability of the digits in a number based on the measurement process and the associated uncertainty. In experimental work, significant figures provide a practical way of ensuring that reported results do not imply a level of precision that is not justified by the measuring instruments.

In a measured quantity, significant figures include all digits that are known with certainty, together with the first uncertain digit. The position of the last significant figure is therefore determined by the uncertainty of the measurement. For example, if a length is reported as

$$x = 4.35 \pm 0.05 \, \text{cm},$$

the digits *4*, *3*, and *5* are significant, while any further digits would not be justified.

The following general rules are used to determine the number of significant figures in a reported value:

- All non-zero digits are significant.

- Zeros between non-zero digits are significant.

- Leading zeros (zeros to the left of the first non-zero digit) are not significant.

- Trailing zeros are significant only if they are explicitly indicated by a decimal point.

Examples:

$$2.35 \quad \text{has three significant figures,}$$
$$0.042 \quad \text{has two significant figures,}$$
$$4.30 \quad \text{has three significant figures,}$$
$$1200 \quad \text{has two significant figures unless stated otherwise.}$$

When performing calculations using measured quantities, the result must be reported with an appropriate number of significant figures. This ensures consistency with the precision of the input data.

For multiplication and division, the result should have the same number of significant figures as the input quantity with the fewest significant figures. For example,

$$2.5 \times 1.24 = 3.10 \ \rightarrow \ 3.1,$$

since 2.5 has only two significant figures.

For addition and subtraction, the result should be reported to the same decimal place as the least precise input value. For example,

$$4.35 + 0.2 = 4.55 \ \rightarrow \ 4.6.$$

In this laboratory course, the number of significant figures reported for a measured quantity must be consistent with its stated uncertainty. The uncertainty determines the position of the last significant figure of the reported value. Reporting more digits than justified by the uncertainty is not acceptable, as it implies a false level of precision.

Throughout this laboratory course, all reported measurements and calculated results are expected to follow these conventions for significant figures. When performing calculations involving measured quantities, rounding according to significant figures should be applied *only to the final result*. Intermediate values produced during calculations, whether by hand or using a calculator, should be kept with extra digits to avoid the accumulation of rounding errors. Premature rounding of intermediate results can lead to incorrect final values and should be avoided.

Consider a simple pendulum of length $L = 6.6$ m with $g = 9.81$ m/s^2. The period is

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Since L has two significant figures, the final result for T must be reported with two significant figures.

Using the correct approach and carrying full precision,

$$T = 2\pi\sqrt{6.6/9.81} = 5.155\ldots\,\text{s} \ \rightarrow \ 5.2\,\text{s}.$$

If the intermediate ratio is rounded prematurely according to significant figures,

$$\frac{L}{g} = 0.67278\ldots \ \rightarrow \ 0.67,$$

then

$$T = 2\pi\sqrt{0.67} = 5.141\ldots\,\text{s} \ \rightarrow \ 5.1\,\text{s}.$$

This example demonstrates that rounding intermediate results—even when done according to significant-figure rules—can shift the final reported value. Therefore, intermediate calculator results should be kept with extra digits, and significant-figure rounding should be applied only once, to the final reported result.

3. Propagation of Uncertainty

In many laboratory experiments, the quantity of interest is not measured directly but is calculated from one or more measured quantities. Since each measured quantity carries an uncertainty, the uncertainty of the calculated result must be determined from the uncertainties of the input quantities. This procedure is known as *propagation of uncertainty.*

The rules of uncertainty propagation presented in this section assume that:

- the uncertainties are small compared to the measured values,

- the measured quantities are independent of each other,

- random errors dominate the uncertainty budget.

Under these conditions, uncertainties can be combined using standard analytical rules.

3.1. General Formula

Let a quantity q be calculated from several measured quantities $x_1, x_2, \ldots, x_n$ according to

$$q = f(x_1, x_2, \ldots, x_n).$$

The uncertainty Δq associated with q is given by

$$\Delta q = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\Delta x_n\right)^2}.$$

This expression forms the basis of all uncertainty propagation rules used in experimental physics.

3.2. Addition and Subtraction

If a quantity is obtained by adding or subtracting measured values,

$$q = a \pm b,$$

the absolute uncertainty in q is

$$\Delta q = \sqrt{(\Delta a)^2 + (\Delta b)^2}.$$

In this case, uncertainties are combined as absolute quantities, not as fractional or percentage uncertainties.

3.3. Multiplication and Division

If a quantity is obtained by multiplying or dividing measured values,

$$q = ab \quad \text{or} \quad q = \frac{a}{b},$$

the fractional uncertainty in q is

$$\frac{\Delta q}{q} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2}.$$

The absolute uncertainty Δq is then obtained by multiplying the fractional uncertainty by the calculated value of q.

3.4. Powers and Products of Powers

If a quantity depends on a measured value raised to a power,

$$q = a^n,$$

the fractional uncertainty is

$$\frac{\Delta q}{q} = |n|\frac{\Delta a}{a}.$$

More generally, if
$$q = a^\alpha b^\beta c^\gamma,$$
the fractional uncertainty in q is
$$\frac{\Delta q}{q} = \sqrt{\left(\alpha \frac{\Delta a}{a}\right)^2 + \left(\beta \frac{\Delta b}{b}\right)^2 + \left(\gamma \frac{\Delta c}{c}\right)^2}.$$

3.5. Example: Simple Pendulum

For a simple pendulum, the gravitational acceleration is determined from
$$g = \frac{4\pi^2 L}{T^2},$$
where L is the pendulum length and T is the period. Using the rules for products and powers, the fractional uncertainty in g is
$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta L}{L}\right)^2 + \left(2\frac{\Delta T}{T}\right)^2}.$$
The absolute uncertainty Δg is then obtained by multiplying this expression by the calculated value of g.

This example illustrates how uncertainties in measured quantities propagate through physical formulas and how certain measurements may contribute more strongly to the final uncertainty.

Reporting the Final Result

The final calculated result should be reported together with its uncertainty,
$$q = q_{\text{calc}} \pm \Delta q,$$
with the uncertainty rounded to one significant figure (or two if the leading digit is 1 or 2). The calculated value q_{calc} should be rounded to the same decimal place as the uncertainty.

Uncertainty propagation should be performed using unrounded intermediate values. Rounding according to significant figures should be applied only to the final reported result.

3.6. Square Root and Logarithmic Functions

Certain mathematical functions appear frequently in laboratory formulas. Two important cases are the square root and the natural logarithm. Their uncertainties can be obtained directly from the general propagation formula.

Square Root

If a quantity q is given by the square root of a measured quantity,
$$q = \sqrt{x},$$
then the fractional uncertainty in q is
$$\frac{\Delta q}{q} = \frac{1}{2}\frac{\Delta x}{x}.$$
Equivalently, the absolute uncertainty is
$$\Delta q = \frac{1}{2}\frac{\Delta x}{x}\, q.$$

This result is a special case of the power-law rule, since $\sqrt{x} = x^{1/2}$. It shows that taking the square root reduces the relative uncertainty by a factor of two.

Natural Logarithm

If a quantity q is given by the natural logarithm of a measured quantity,

$$q = \ln x,$$

then the absolute uncertainty in q is

$$\Delta q = \frac{\Delta x}{x}.$$

In this case, the uncertainty of $\ln x$ depends only on the fractional uncertainty of x. Since the natural logarithm is a dimensionless function, its uncertainty is also dimensionless.

General Logarithms

For a logarithm with an arbitrary base b,

$$q = \log_b x,$$

the uncertainty is

$$\Delta q = \frac{1}{\ln b} \frac{\Delta x}{x}.$$

Remarks

These results emphasize the importance of fractional uncertainty. Functions such as square roots and logarithms do not depend on the absolute size of the uncertainty alone, but on how large the uncertainty is relative to the measured value itself. Consequently, quantities with small fractional uncertainty lead to more reliable results when used inside nonlinear functions.

Remark on the Use of Propagation Formulas

All specific uncertainty propagation rules presented in this section—such as those for sums, products, powers, square roots, and logarithmic functions—are not independent formulas. They are direct consequences of the general uncertainty propagation formula given at the beginning of this section.

Students are strongly encouraged to verify each special case by starting from the general formula and computing the required partial derivatives explicitly. Doing so reinforces the understanding that uncertainty propagation is a systematic procedure based on the functional dependence of a quantity on measured variables, rather than a collection of formulas to be memorized.

Whenever a formula is encountered that does not match any of the standard cases, the general propagation formula should be used directly.

Important Assumption on Independence of Variables

Throughout this laboratory course, all measured quantities are assumed to be statistically independent of each other. Under this assumption, their uncertainties can be combined using the general uncertainty propagation formula presented at the beginning of this section, as well as all the specific propagation rules derived from it.

If measured quantities are not independent—for example, if they are correlated due to the measurement procedure or shared systematic effects—the propagation formulas given in this section are no longer valid in their present form and must be modified to include correlation terms. Since the general propagation formula itself is derived under the assumption of independence, all results presented here rely on this assumption from the outset.

In this course, correlated uncertainties will not be treated. Students should therefore always verify that the quantities being combined can reasonably be regarded as independent before applying the uncertainty propagation formulas.

4. Repeated Measurements, Mean Value, and Statistical Uncertainty

In experimental physics, a single measurement rarely provides complete information about the reliability of a result. When the same physical quantity is measured repeatedly under identical conditions, the measured values usually differ

slightly from one another. These differences arise naturally from random influences that cannot be completely eliminated, such as small environmental fluctuations, limitations of human perception, and intrinsic noise in measuring instruments. Repeated measurements therefore provide insight not only into the value of a physical quantity, but also into the nature and magnitude of the associated uncertainty.

When a measurement is repeated many times, the resulting values typically form a characteristic bell-shaped distribution centered around a central value. This type of distribution is known as a *Gaussian* or *normal* distribution. Its detailed mathematical form is not required here, but its qualitative features are essential:

- most measured values lie close to the central value,
- large deviations from the center occur less frequently,
- the distribution is approximately symmetric.

This behavior is illustrated schematically in Fig. 3. The horizontal axis represents student scores, while the vertical axis represents the number of students obtaining a given score. Most students cluster around the average score, while very low and very high scores occur less frequently, producing the characteristic bell-shaped curve.
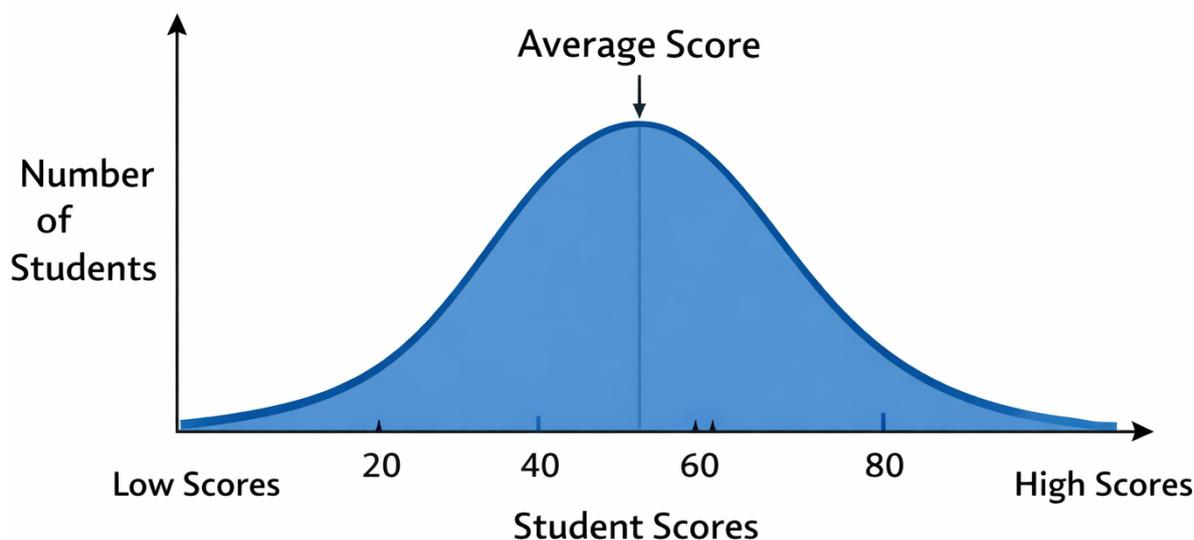

Figure 3: Schematic bell-shaped (Gaussian) distribution of student scores. Most values cluster around the average score, while extreme low and high scores occur less frequently. Similar distributions arise naturally in repeated experimental measurements due to random effects.

This behavior arises because each measurement is influenced by many small, independent random effects. According to a fundamental statistical principle known as the *central limit theorem*, the combined effect of such independent random contributions naturally leads to a Gaussian distribution, regardless of the details of the individual effects.

4.1. Mean Value as Best Estimate

Suppose a physical quantity x is measured N times, producing the values

$$x_1, x_2, \ldots, x_N.$$

The most reliable estimate of the measured quantity is given by the *arithmetic mean*,

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

The mean value corresponds to the center of the measurement distribution and represents the best estimate of the quantity when random errors dominate. As the number of measurements increases, the mean value becomes increasingly stable, even though individual measurements may continue to fluctuate.

4.2. Scatter of Measurements and Precision

While the mean value characterizes the central tendency of the measurements, the spread of individual values around the mean reflects the *precision* of the measurement process. This spread is quantified by the *standard deviation*,

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}.$$

The standard deviation provides a measure of the typical deviation of individual measurements from the mean. A small standard deviation indicates high precision and good reproducibility, whereas a large standard deviation indicates significant random variability.

Although individual measurements may be widely scattered, their average is usually determined more accurately. The uncertainty associated with the mean value is given by

$$\Delta\bar{x} = \frac{\sigma}{\sqrt{N}}.$$

This quantity represents the uncertainty in the estimated mean due to random errors. As the number of measurements increases, the uncertainty of the mean decreases, reflecting the improved reliability obtained through averaging. This reduction of uncertainty is another direct consequence of the central limit theorem and illustrates why repeated measurements are a powerful tool in experimental physics.

Consider a length measured five times using the same meter stick, yielding the following results (in centimeters):

$$4.34,\ 4.36,\ 4.35,\ 4.33,\ 4.37.$$

The mean value is

$$\bar{x} = 4.35\,\text{cm}.$$

The standard deviation of the measurements is

$$\sigma = 0.016\,\text{cm},$$

and the uncertainty of the mean is

$$\Delta\bar{x} = \frac{0.016}{\sqrt{5}} \approx 0.007\,\text{cm}.$$

After rounding according to the conventions of this laboratory course, the result may be reported as

$$x = 4.35 \pm 0.01\,\text{cm}.$$

4.3. Weighted Mean as the Best Estimate

The arithmetic mean is appropriate when all measurements have the same uncertainty. However, in many experiments individual measurements may have different uncertainties. In such cases, measurements with smaller uncertainty should contribute more strongly to the final estimate.

Suppose a physical quantity x is measured N times, producing the values

$$x_1, x_2, \ldots, x_N,$$

with associated standard uncertainties

$$\sigma_1, \sigma_2, \ldots, \sigma_N.$$

To account for the differing precisions, we assign each measurement a weight

$$w_i = \frac{1}{\sigma_i^2}.$$

Measurements with smaller uncertainty therefore receive larger weights.

The best estimate of the quantity is then given by the *weighted mean*,

$$\bar{x}_w = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}.$$

If all uncertainties are equal, then all weights are equal and the weighted mean reduces to the ordinary arithmetic mean.

The uncertainty associated with the weighted mean is given by

$$\Delta \bar{x}_w = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i}}.$$

This expression shows that the overall precision increases as the total weight increases. Measurements with smaller uncertainty therefore reduce the final uncertainty more effectively.

4.4. Example: Weighted Mean of Resistance Measurements

Suppose a resistance is obtained from three independent determinations, either by using different instruments or by evaluating $R = V/I$ from separate voltage–current measurements, yielding

$$R_1 = 10.12 \pm 0.05 \,\Omega,$$
$$R_2 = 10.08 \pm 0.02 \,\Omega,$$
$$R_3 = 10.15 \pm 0.10 \,\Omega.$$

Step 1: Calculate the weights

$$w_1 = \frac{1}{(0.05)^2} = 400, \qquad w_2 = \frac{1}{(0.02)^2} = 2500, \qquad w_3 = \frac{1}{(0.10)^2} = 100.$$

Step 2: Compute the weighted mean

$$\bar{R}_w = \frac{400(10.12) + 2500(10.08) + 100(10.15)}{400 + 2500 + 100}.$$

$$\bar{R}_w = \frac{4048 + 25200 + 1015}{3000} = 10.09 \,\Omega.$$

Step 3: Compute the uncertainty of the weighted mean

$$\Delta \bar{R}_w = \sqrt{\frac{1}{400 + 2500 + 100}} = \sqrt{\frac{1}{3000}} = 0.018 \,\Omega.$$

After rounding according to laboratory conventions, the final result is reported as

$$R = 10.09 \pm 0.02 \,\Omega.$$

This example illustrates that measurements with smaller uncertainty contribute more strongly to the final estimate and reduce the overall uncertainty.

Limitations and Practical Remarks

The statistical uncertainty obtained from repeated measurements reflects only the effects of random errors. Systematic errors, such as instrument miscalibration or consistent experimental bias, are not revealed by the scatter of repeated measurements and are not reduced by averaging.

In this laboratory course, students should perform repeated measurements whenever possible, examine the distribution of results, and **report** the mean value together with an appropriate uncertainty, while remaining aware of the limitations imposed by systematic effects.

5. Data Representation and Graphs

A central goal of experimental physics is to test relationships between physical quantities. Many physical laws predict that one quantity depends on another in a simple and specific way, such as proportionality, power-law behavior, or exponential dependence. An effective way to examine such relationships is through graphical representation of experimental data.

Graphs provide a clear visual method for comparing experimental results with theoretical expectations. Patterns that may be difficult to recognize from numerical tables alone often become immediately apparent when data are plotted. For this reason, graphs play a fundamental role in the analysis and interpretation of experimental measurements.

If a physical law predicts that one quantity y is proportional to another quantity x, then their relationship may be written as

$$y \propto x.$$

In this case, a graph of y plotted against x should yield a straight line passing through the origin. Because straight lines are easy to recognize visually, this provides a simple and effective test of proportionality.

In practice, experimental data points rarely lie exactly on a perfect straight line due to measurement uncertainties. The key question is whether the observed deviations from linearity can be reasonably attributed to these uncertainties or whether they indicate a failure of the assumed physical relationship.

As we discussed, measured quantities are always subject to uncertainty, and this must be taken into account when interpreting graphs. Each plotted point represents not an exact value, but a range of possible values consistent with the measurement uncertainty. These ranges can be represented graphically using error bars.

Not all physical relationships are linear. In many cases, one quantity depends on another through a power law or an exponential function. Directly plotting such data may produce a curved graph, making visual assessment difficult.

A common and powerful technique is to transform the variables so that the expected relationship becomes linear. For example:

- If y is proportional to a power of x, plotting y against an appropriate power of x may yield a straight line.

- If y depends exponentially on x, plotting the natural logarithm of y against x should produce a straight line.

Such linearization techniques allow nonlinear relationships to be analyzed using the same straightforward graphical methods as linear ones. In this laboratory course, students are encouraged to represent experimental data graphically whenever appropriate, to include uncertainties in their plots, and to use graphs as a primary means of assessing the validity of physical relationships.

The following examples demonstrate how graphs are used to examine relationships between physical quantities in laboratory experiments. One example illustrates a *linear relationship*, in which measured data are plotted directly and compared with a straight-line trend. The second example illustrates a *nonlinear relationship*, which is first transformed into a linear form through an appropriate choice of variables.

In both cases, experimental uncertainties are taken into account, and graphical analysis is used at this stage as a qualitative tool. A more quantitative treatment using least-squares fitting will be introduced later in the course.

5.1. Example: Testing a Linear Law with a Graph (Ohm's Law)

Many experiments in this laboratory course are designed to test whether two quantities are related by a linear law. A simple and important example is *Ohm's law*, which states that the potential difference V across a conductor is proportional to the current I through it:

$$V = RI,$$

where R is the (constant) resistance of the conductor. If Ohm's law holds, a graph of V (vertical axis) against I (horizontal axis) should be a straight line passing through the origin, and the slope of the line is the resistance R.

To illustrate this method, suppose we measure the voltage across a resistor for several values of current. For simplicity, assume that the current values are known very accurately (their uncertainty is negligible compared with the voltage uncertainty), while each voltage reading has an uncertainty of $\pm 0.02\,\mathrm{V}$. A possible data set is shown in Table 4.

The values from Table 4 are plotted in Fig. 4. Because the uncertainty in I is negligible, each point is shown with a *vertical* error bar only. For example, the first measurement $(I, V) = (0.10\,\mathrm{A},\ 0.94\,\mathrm{V})$ represents a voltage that likely lies in the interval $0.94 \pm 0.02\,\mathrm{V}$, so the corresponding point on the graph lies somewhere on the vertical segment at $I = 0.10\,\mathrm{A}$.

Table 4: Current and voltage measurements used to examine the linear relationship between V and I. The uncertainty in the current is negligible, while all voltage measurements have an uncertainty of $\pm 0.20\,\text{V}$.

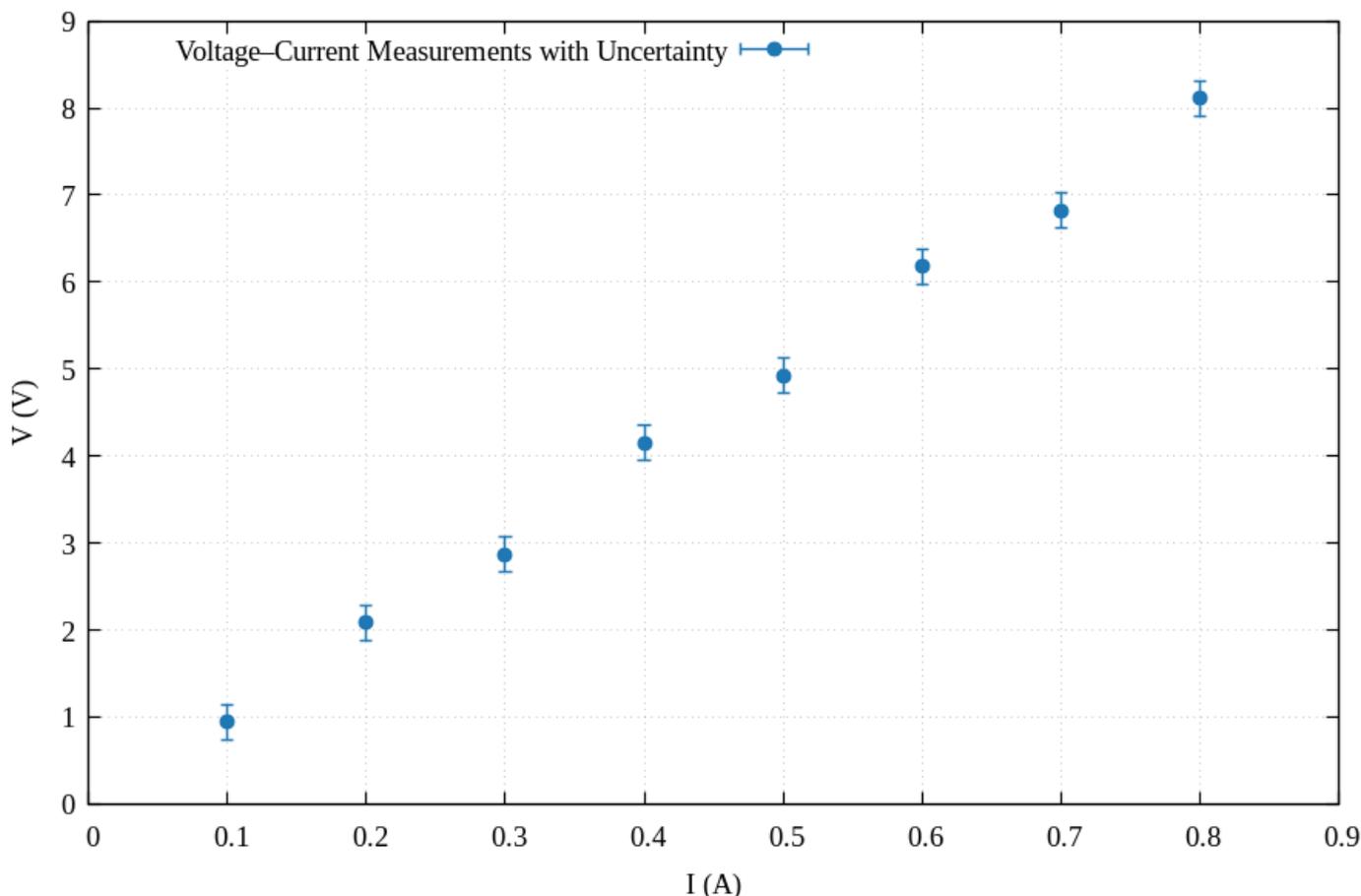| Current I (A)	Voltage V (V)	Uncertainty in V (V)
0.10	0.94	± 0.20
0.20	2.08	± 0.20
0.30	2.87	± 0.20
0.40	4.15	± 0.20
0.50	4.92	± 0.20
0.60	6.18	± 0.20
0.70	6.82	± 0.20
0.80	8.11	± 0.20

Figure 4: Graph of voltage V versus current I using the data from Table 4. Only the V-values have appreciable uncertainty, so vertical error bars are shown. A straight line through the origin consistent with the error bars supports Ohm's law, and its slope gives the resistance R.

When a straight line consistent with the expected physical relationship passes through, or reasonably close to, the error bars of most data points, the data may be regarded as consistent with the proposed relationship. Conversely, if the data systematically deviate from the expected trend beyond the uncertainty ranges, this suggests the presence of experimental errors, incorrect assumptions, or a breakdown of the physical model.

Graphs are not only useful for testing relationships, but also for determining physical constants. When a linear relationship is expected, the slope of the best-fit line often corresponds directly to a physical parameter of interest. By estimating the range of slopes that remain consistent with the experimental uncertainties, one can also estimate the uncertainty in the determined parameter.

If Ohm's law is valid for the resistor in the range tested, we should be able to draw a straight line through the origin that passes through, or reasonably close to, most of the vertical error bars. The slope of this best-fit line gives the resistance R. If a straight line through the origin systematically misses many error bars by a distance comparable to (or larger than) the error-bar length, then the data are not consistent with a simple proportionality $V \propto I$, and the measurement procedure, uncertainty estimates, or the physical assumptions must be reconsidered.

5.2. Example: Linearization of the Simple Pendulum

A common example of a nonlinear relationship encountered in the laboratory is the period of a simple pendulum. For small oscillation amplitudes, the period T of a pendulum of length L is given by

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the acceleration due to gravity. This equation shows that T is not linearly proportional to L, but rather to the square root of L.

Because visually assessing whether data follow a square-root dependence is difficult, it is advantageous to rewrite the relation in a linear form. Squaring both sides gives

$$T^2 = \frac{4\pi^2}{g} L.$$

This predicts a linear relationship between T^2 and L. Therefore, a graph of T^2 (vertical axis) against L (horizontal axis) should yield a straight line passing through the origin, within experimental uncertainties.

Suppose the period T is measured for several pendulum lengths L. For each length, the measured period is recorded together with its uncertainty ΔT. Since the graph uses T^2, we must calculate T^2 for each measurement and also determine the uncertainty $\Delta(T^2)$ using uncertainty propagation.

A possible set of measurements is shown in Table 5. In this example, the uncertainty in length is assumed to be negligible compared with the uncertainty in timing, so only vertical uncertainty bars will be used in the graph. In this example, the period T is measured using a stopwatch with a timing uncertainty of $\Delta T = 0.05$ s for all measurements.

Table 5: Example data for the linearized pendulum relation. The measured quantities are the pendulum length L and the period T. All period measurements have an uncertainty $\Delta T = 0.05$ s. The plotted quantity is T^2, and its uncertainty is obtained from $\Delta(T^2) = 2T\Delta T$. The uncertainty in L is assumed to be negligible.

Length L (m)	Period T (s)	T^2 (s^2)	$\Delta(T^2)$ (s^2)
0.30	1.10	1.210	0.110
0.40	1.29	1.664	0.129
0.50	1.42	2.016	0.142
0.60	1.55	2.403	0.155
0.70	1.67	2.789	0.167
0.80	1.80	3.240	0.180
0.90	1.90	3.610	0.190
1.00	2.02	4.080	0.202

Since the plotted quantity is T^2, the uncertainty in T must be propagated. Using the general uncertainty propagation principles discussed earlier, for

$$y = T^2,$$

the uncertainty in y is

$$\Delta(T^2) = 2T\,\Delta T.$$

Thus, each point on the graph corresponds to a measured value of T^2 at a given L, together with a vertical uncertainty bar of length $\pm\Delta(T^2)$.

The data listed in Table 5 are plotted in Fig. 5 as T^2 versus L, with vertical error bars representing the propagated uncertainty $\Delta(T^2)$. Each plotted point therefore corresponds not to a single exact value, but to a range of values consistent with the timing uncertainty of the measurement.

It is also important to note that the size of the vertical error bars increases with increasing period T, and therefore with increasing T^2. This behavior is not accidental, but follows directly from the uncertainty propagation. Since the uncertainty in the plotted quantity is given by $\Delta(T^2) = 2T\,\Delta T$, a constant timing uncertainty ΔT leads to larger uncertainties in T^2 for longer periods.

This systematic increase in the uncertainty bars is clearly visible in Fig. 5 and provides a concrete illustration of how measurement uncertainties propagate through nonlinear transformations of measured quantities.
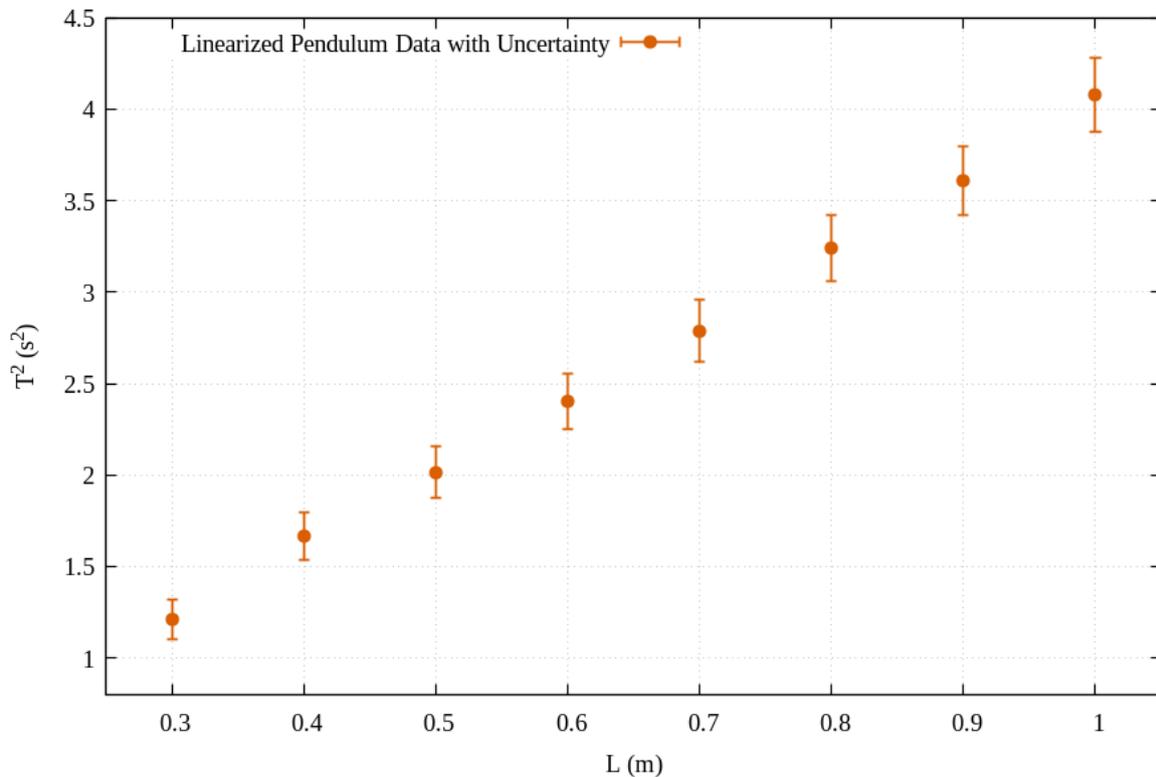
Figure 5: Graph of T^2 versus pendulum length L using the data from Table 5. Vertical error bars represent the propagated uncertainty in T^2.

As shown in Fig. 5, the data points and their associated uncertainty bars are broadly consistent with a straight-line relationship between T^2 and L. This visual agreement supports the predicted dependence of the pendulum period on length for small oscillation amplitudes. At this stage, the graph is used only for qualitative assessment; a quantitative determination of the slope and its uncertainty will be carried out later using least-squares fitting.

The slope of the line is

$$\text{slope} = \frac{4\pi^2}{g},$$

so a measurement of the slope allows g to be determined. At this stage of the course, the straight line is assessed qualitatively by visual inspection with uncertainties in mind. A systematic determination of the best-fit line and the uncertainty in its slope will be carried out later, after the method of least squares has been introduced.

6. Drawing Graphs on Graph Paper

Drawing graphs by hand on graph paper is an essential experimental skill. It helps students understand how numerical data, uncertainties, and physical relationships are represented geometrically. In this section, the procedure for drawing graphs on graph paper is explained step by step, using schematic examples.

6.1. Graph Paper and Scale

Graph paper consists of evenly spaced squares. Typically, a darker line marks every fifth square, forming a larger square. These large squares are used to define the numerical scale of the axes.

Before plotting any data, a scale must be chosen for each axis. The scale determines how physical quantities are mapped onto distances on the paper. Each large square is typically divided into five minor squares in each direction. The minor squares allow intermediate values to be placed accurately between the major scale markings. Once chosen, the scale must be used consistently across the entire graph.

For example, a student may decide that:

- one large square on the horizontal axis represents $0.10\,\text{m}$, so one minor square represents $0.02\,\text{m}$,

- one large square on the vertical axis represents $0.50\,\text{s}^2$, so one minor square represents $0.10\,\text{s}^2$.
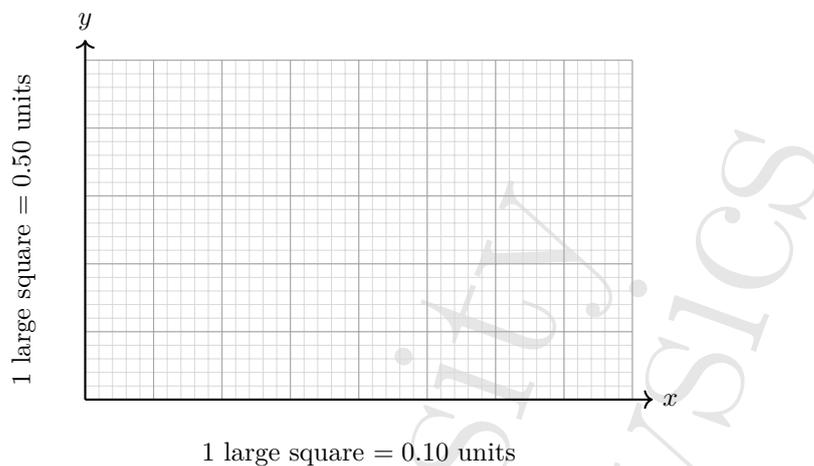
17

Figure 6: Graph-paper style grid showing major and minor squares. The numerical scale is defined using the large squares.

6.2. Placing a Data Point

Once the scale is fixed, numerical values can be translated into positions on the graph paper. This can be done in two equivalent ways. The first method is to divide the numerical value of the data point by the value represented by one minor square, which directly gives the number of minor squares to be counted from the origin. Alternatively, the position of the point can be determined by visual inspection using the major and minor grid lines, which with practice is often faster and more convenient.

Suppose a measurement yields the values

$$x = 0.35, \qquad y = 1.40.$$

Using the scale shown in Fig. 7, the student proceeds as follows:

- $x = 0.35$ corresponds to 3.5 large squares by inspection, or equivalently to 17.5 minor squares obtained from $0.35/0.02$, along the horizontal axis;

- $y = 1.40$ corresponds to 14 minor squares, either by inspection or from the calculation $1.40/0.10$, along the vertical axis.

The data point is then placed at the intersection of these two positions on the graph paper.
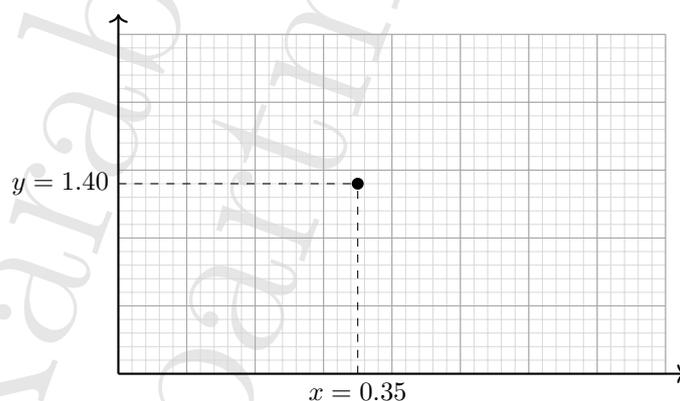

Figure 7: Placement of a data point using the chosen scale. Dashed guide lines help locate the correct position.

With practice, students will develop their own efficient and reliable way of reading scales, counting squares, and placing points accurately, and drawing graphs by hand will become a natural and intuitive part of experimental work. Students are encouraged to practice drawing graphs regularly, as experience quickly improves both accuracy and confidence in placing data points by hand.

If a measurement has an associated uncertainty, this must be represented by an error bar. An error bar shows the range within which the true value of the measured quantity is expected to lie.

For example, suppose the uncertainty in the vertical-axis quantity is ±0.20, this corresponds to 2 minor squares (by inspection) on the vertical axis. The half-length of the error bar is then 2 minor squares. The error bar is therefore drawn symmetrically two minor squares above and two minor squares below the plotted point, using the same scale as the axis.
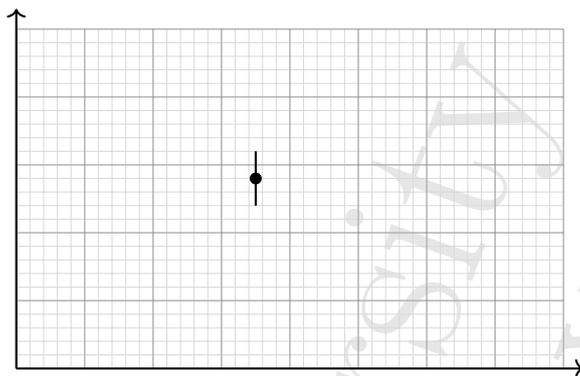

Figure 8: A plotted data point with a vertical error bar drawn using the same scale as the axis.

6.3. General Remarks

When drawing graphs on graph paper:

- choose simple, uniform scales before plotting any points,

- use the grid to convert numerical values into distances accurately,

- plot points carefully without adjusting them to fit an expected trend,

- draw error bars using the same scale as the corresponding axis.

A well-drawn graph should clearly show the data, the scale, and the uncertainties, allowing the experimental results to be interpreted without referring back to the raw data table.

7. Least-Squares Fit and Uncertainty of the Fitted Parameters

In the previous sections, data were plotted on graph paper and examined visually. While visual inspection allows us to judge whether the data are consistent with a straight line, it does not provide a quantitative and objective determination of the slope and intercept.

To determine the optimal straight line and its associated uncertainty, we use the method of least squares.

Assume a linear relationship

$$y = mx + b,$$

and suppose we have N measured data points (x_i, y_i). Each y_i has an associated uncertainty σ_i, represented by a vertical error bar.

If the uncertainties are known, the appropriate quantity to minimize is

$$\chi^2 = \sum_{i=1}^{N} \frac{[y_i - (mx_i + b)]^2}{\sigma_i^2}.$$

Define the weights

$$w_i = \frac{1}{\sigma_i^2}.$$

Define the common quantity

$$\Delta = \left(\sum w_i\right)\left(\sum w_i x_i^2\right) - \left(\sum w_i x_i\right)^2.$$

Then the fitted slope and intercept are

$$m = \frac{\left(\sum w_i\right)\left(\sum w_i x_i y_i\right) - \left(\sum w_i x_i\right)\left(\sum w_i y_i\right)}{\Delta},$$

$$b = \frac{(\sum w_i x_i^2)(\sum w_i y_i) - (\sum w_i x_i)(\sum w_i x_i y_i)}{\Delta}.$$

The uncertainties of the fitted parameters are

$$\Delta m = \sqrt{\frac{\sum w_i}{\Delta}}, \qquad \Delta b = \sqrt{\frac{\sum w_i x_i^2}{\Delta}}.$$

If all uncertainties are equal, then all weights are equal. In that case one may set $w_i = 1$ to compute m and b. (The parameter uncertainties must still be expressed in physical units using the known common uncertainty; see the Ohm's law example below.)

Example: Ohm's Law (Unweighted Fit; Equal Uncertainties)

In this experiment we take

$$x = I, \qquad y = V,$$

and we assume that all voltage measurements have the *same* uncertainty

$$\sigma_V = \sigma \qquad \text{(constant for all points)}.$$

When all σ_i are equal, the weights $w_i = 1/\sigma_i^2$ are also equal. In this case, the common factor w *cancels* in the expressions for the fitted parameters m and b, so the best-fit line can be obtained using the ordinary (unweighted) sums. However, the same factor does *not* cancel in the parameter uncertainties; instead it factors out as the common uncertainty σ. Later, one may set $w = 1/\sigma^2$ explicitly to connect with the weighted formulas.

Table 6: Ohm's law data and the common terms needed for the least-squares sums. Here $x = I$ and $y = V$.

I (A)	V (V)	I^2 (A^2)	IV (AV)
0.10	0.94	0.010	0.094
0.20	2.08	0.040	0.416
0.30	2.87	0.090	0.861
0.40	4.15	0.160	1.660
0.50	4.92	0.250	2.460
0.60	6.18	0.360	3.708
0.70	6.82	0.490	4.774
0.80	8.11	0.640	6.488
$\sum I = 3.600$	$\sum V = 36.07$	$\sum I^2 = 2.040$	$\sum IV = 20.461$

Define the common denominator (often written as Δ):

$$\Delta = N \sum I^2 - \left(\sum I\right)^2.$$

With $N = 8$,

$$\Delta = 8(2.040) - (3.600)^2 = 3.36.$$

Best-fit parameters (unweighted). The slope and intercept are

$$m = \frac{N \sum IV - (\sum I)(\sum V)}{\Delta} = \frac{8(20.461) - (3.600)(36.07)}{3.36} = 10.07,$$

$$b = \frac{(\sum I^2)(\sum V) - (\sum I)(\sum IV)}{\Delta} = \frac{(2.040)(36.07) - (3.600)(20.461)}{3.36} = -0.0229.$$

Thus the best-fit line is

$$V = mI + b = (10.07)\,I - 0.023.$$

Uncertainties of the fitted parameters (equal σ). When all y_i have the same standard uncertainty σ, the parameter uncertainties are

$$\sigma_m = \sigma \sqrt{\frac{N}{\Delta}}, \qquad \sigma_b = \sigma \sqrt{\frac{\sum I^2}{\Delta}}.$$

Using $\sigma = \sigma_V = 0.20$ V,

$$\sigma_m = 0.20\sqrt{\frac{8}{3.36}} = 0.31 \ \Omega, \qquad \sigma_b = 0.20\sqrt{\frac{2.040}{3.36}} = 0.16 \text{ V}.$$

Final reported result.

$$V = (10.07 \pm 0.31)\,\Omega \cdot I + (-0.02 \pm 0.16)\,\text{V},$$

and the resistance is

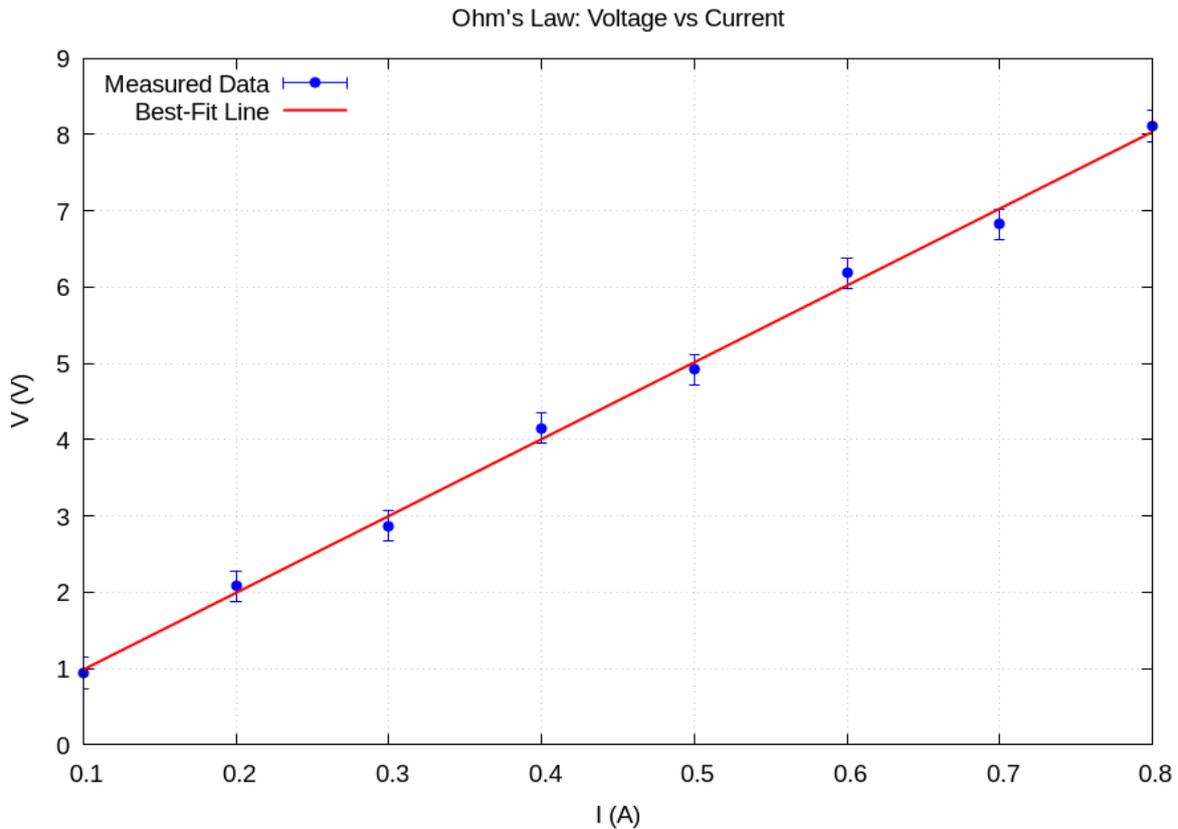$$R = (10.07 \pm 0.31)\,\Omega.$$

Figure 9: Voltage V versus current I for the Ohm's law experiment. The blue points represent the measured data with vertical error bars ($\Delta V = 0.20$ V). The red line shows the best-fit straight line obtained from least-squares analysis.

Figure 9 shows the measured data together with the fitted line. The small intercept relative to its uncertainty confirms that the data are consistent with direct proportionality within experimental precision.

Example: Linearized Pendulum (Unequal Uncertainties; Weighted Fit)

We linearize

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \Longrightarrow \quad T^2 = \frac{4\pi^2}{g}L.$$

Take

$$x = L, \qquad y = T^2, \qquad \Delta T = 0.05 \text{ s}.$$

The plotted uncertainty is obtained from propagation:

$$\Delta(T^2) = 2T\,\Delta T.$$

Since $\Delta(T^2)$ differs from point to point, we must use weighted least squares with

$$w_i = \frac{1}{\sigma_i^2} = \frac{1}{[\Delta(T^2)_i]^2}.$$

Table 7: Pendulum linearization data and the common weighted terms. Here $x = L$, $y = T^2$, $\sigma_i = \Delta(T^2)$, and $w_i = 1/\sigma_i^2$.

L (m)	T (s)	T^2 (s^2)	$\Delta(T^2)$ (s^2)	w	wL	wT^2	wL^2	wLT^2
0.30	1.10	1.210	0.110	82.645	24.793	100.000	7.438	30.000
0.40	1.29	1.664	0.129	60.093	24.037	99.994	9.615	39.998
0.50	1.42	2.016	0.142	49.593	24.797	99.980	12.398	49.990
0.60	1.55	2.403	0.155	41.623	24.974	100.021	14.984	60.012
0.70	1.67	2.789	0.167	35.856	25.100	100.004	17.570	70.003
0.80	1.80	3.240	0.180	30.864	24.691	100.000	19.753	80.000
0.90	1.90	3.610	0.190	27.701	24.931	100.000	22.438	90.000
1.00	2.02	4.080	0.202	24.507	24.507	99.990	24.507	99.990
$\sum$				352.883	197.830	799.989	128.703	519.993

Using the sums from Table 7:

$$\Delta = \left(\sum w\right)\left(\sum wL^2\right) - \left(\sum wL\right)^2 = (352.883)(128.703) - (197.830)^2 = 6280.45.$$

Hence,

$$m = \frac{\left(\sum w\right)\left(\sum wLT^2\right) - \left(\sum wL\right)\left(\sum wT^2\right)}{\Delta} = 4.018,$$

$$b = \frac{\left(\sum wL^2\right)\left(\sum wT^2\right) - \left(\sum wL\right)\left(\sum wLT^2\right)}{\Delta} = 0.0145.$$

The parameter uncertainties are

$$\Delta m = \sqrt{\frac{\sum w}{\Delta}} = 0.237, \qquad \Delta b = \sqrt{\frac{\sum wL^2}{\Delta}} = 0.143.$$

Therefore,

$$T^2 = (4.02 \pm 0.24)\, L + (0.01 \pm 0.14).$$

From theory,

$$m = \frac{4\pi^2}{g} \quad \Longrightarrow \quad g = \frac{4\pi^2}{m}.$$

Propagating uncertainty from m gives

$$\Delta g = \left|\frac{dg}{dm}\right| \Delta m = \frac{4\pi^2}{m^2}\, \Delta m = g\, \frac{\Delta m}{m}.$$

Numerically,

$$g = 9.83 \text{ m/s}^2, \qquad \Delta g = 0.58 \text{ m/s}^2,$$

so

$$g = (9.8 \pm 0.6)\, \text{m/s}^2.$$

Figure 10 illustrates the linearization of the pendulum relation. Because the uncertainties in T^2 increase with T, the data points do not all have equal weight. The weighted least-squares method therefore ensures that measurements with smaller propagated uncertainty contribute more strongly to the determination of the slope.
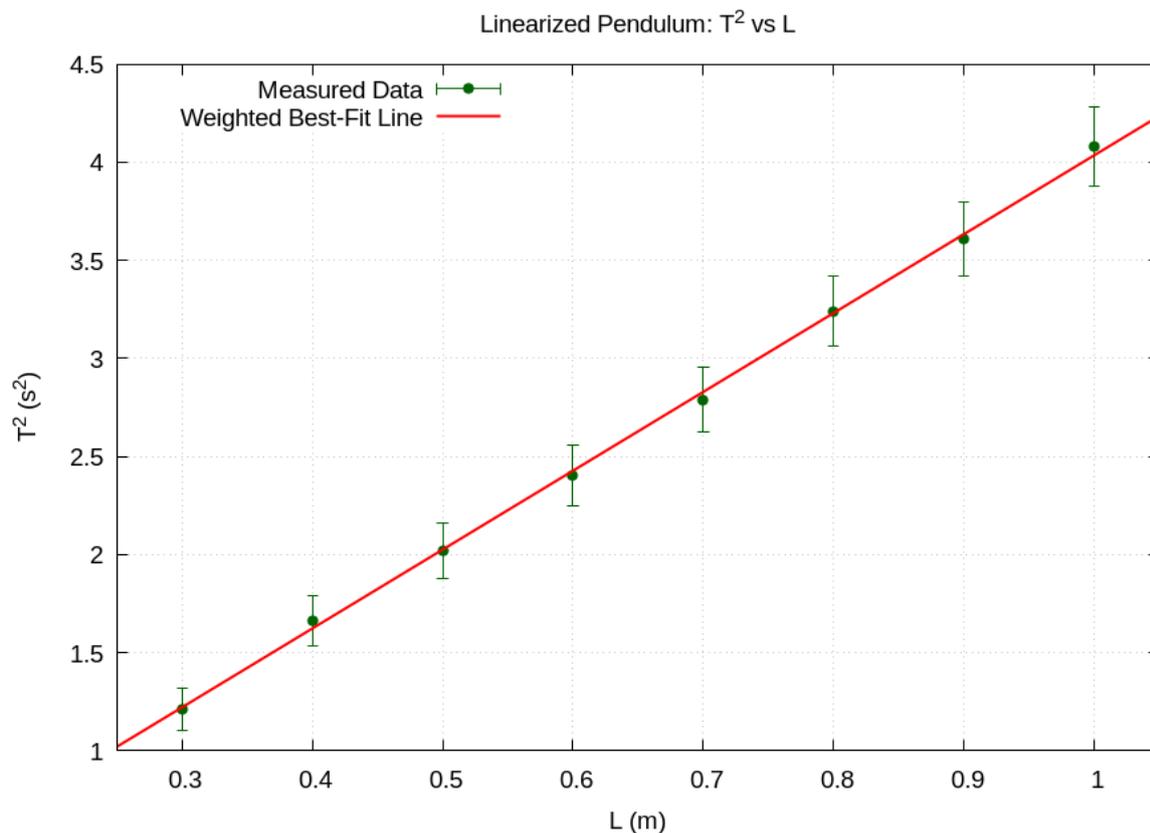
Figure 10: Linearized pendulum data: T^2 plotted against L. The vertical error bars represent the propagated uncertainties $\Delta(T^2) = 2T\Delta T$. The red line shows the weighted least-squares fit.

Acknowledgments

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